Belief models

A very general theory of aggregation

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My plan is to show how far we can get with just these abstract ideas.

Introduction (again)

The very general theory of "Belief Models"[1] provides a neat generalisation of (part of) AGM belief revision theory.

[1] Gert de Cooman. "Belief models: An order-theoretic investigation". *Annals of Mathematics and Artificial Intelligence* 45 (2005), pp. 5–34

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Introduction (again)

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My plan is to show that the same sort of generalisation can be applied to "merging operators"[2] for aggregating (propositional) knowledge bases.

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Belief models

The recipe

- AGM expansion
- AGM revision
- Merging operators

Cooking up aggregation rules

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Top The set of all sentences – the top of the ordering – is not coherent.

Lower previsions

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Lower probabilities (lower previsions restricted to events) are superadditive but not necessarily additive:
$L(X \text{ or} Y) \geq L(X) + L(Y)$ for incompatible X, Y.

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Top The lower prevision that assigns ∞ to all gambles – the top of the structure – is not coherent.

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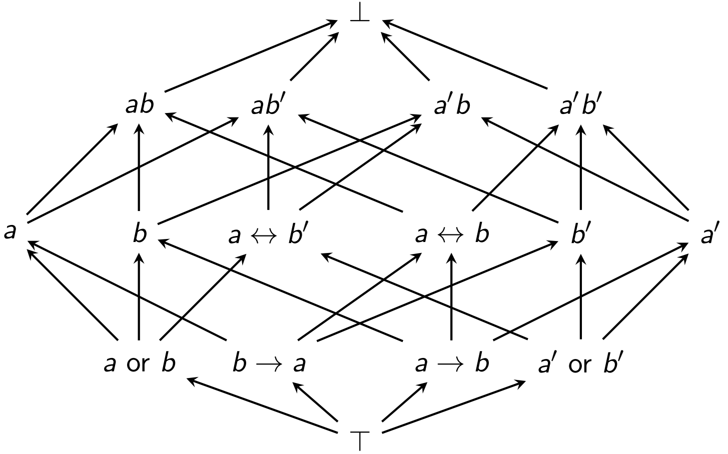
$\langle \mathbf{S}, \mathbf{C}, \preceq \rangle$ is called a *belief structure*.

Lattice structure



Closure

Let $\overline{\mathbf{C}} = \mathbf{C} \cup \{1_{\mathbf{S}}\}$, and define:

$$Cl_{\mathbf{S}}(b) = \inf\{c \in \overline{\mathbf{C}}, b \preceq c\}$$

Closure for sets of sentences

$$\{A, B, C, A \wedge B, \neg(A \wedge B) \rightarrow A \wedge B, \ldots\}$$

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- Sets of desirable gambles, choice functions...
- Preference relations, comparative confidence relations?

AGM basics

We have a propositional logic $\mathcal{L}$, and use a set of sentences K to represent the beliefs of an agent. The agent beliefs $X \in \mathcal{L}$ just in case $X \in K$.

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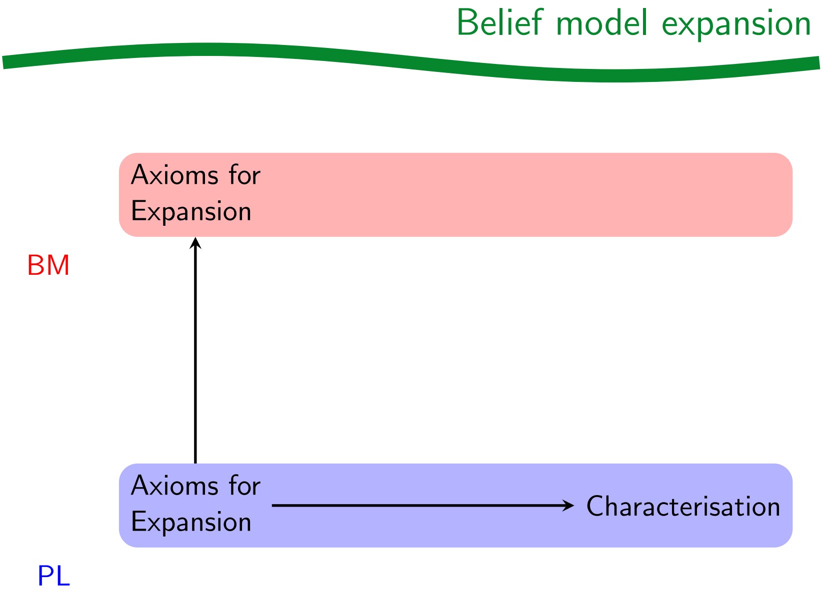
Of particular interest are those agents whose belief set K is consistent, and closed under entailment.

We can provide some axioms for straightforward learning A given belief set K, such that K^+_A can be characterised.

Belief model expansion

Axioms for Expansion → Characterisation

PL


Belief model expansion
Axioms for Expansion
BM
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Characterisation
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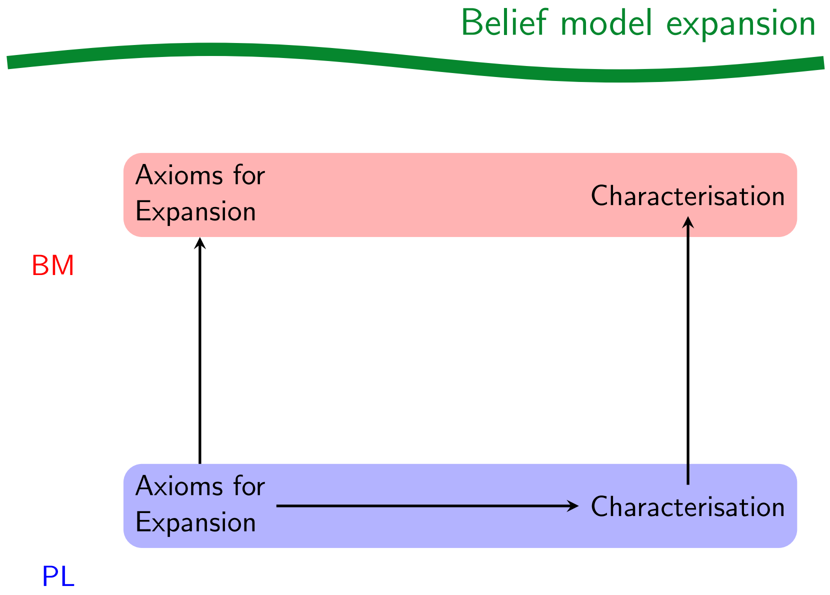
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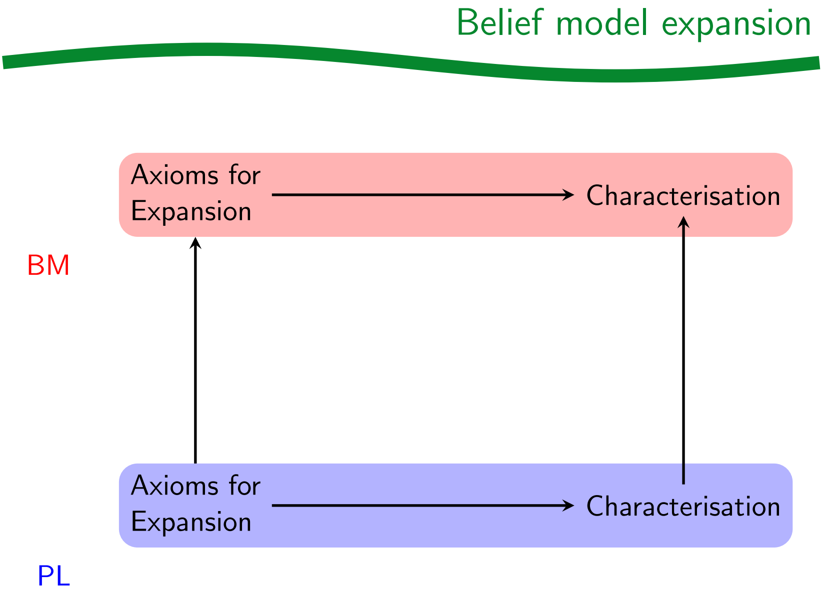
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The recipe

This recipe is quite generalisable: take a result framed in the theory of propositional logic, and (if you're lucky) it will also hold in some version of the belief models framework.

Strong belief structures

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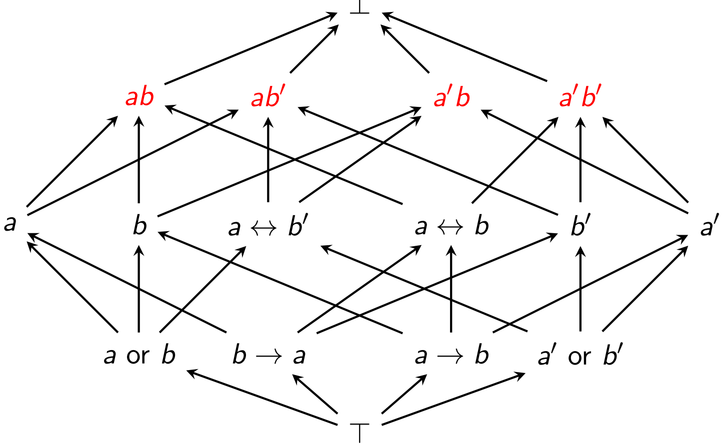
I suspect that this property can be weakened, but that is future work.

Lattice structure



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Revision

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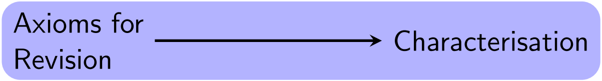
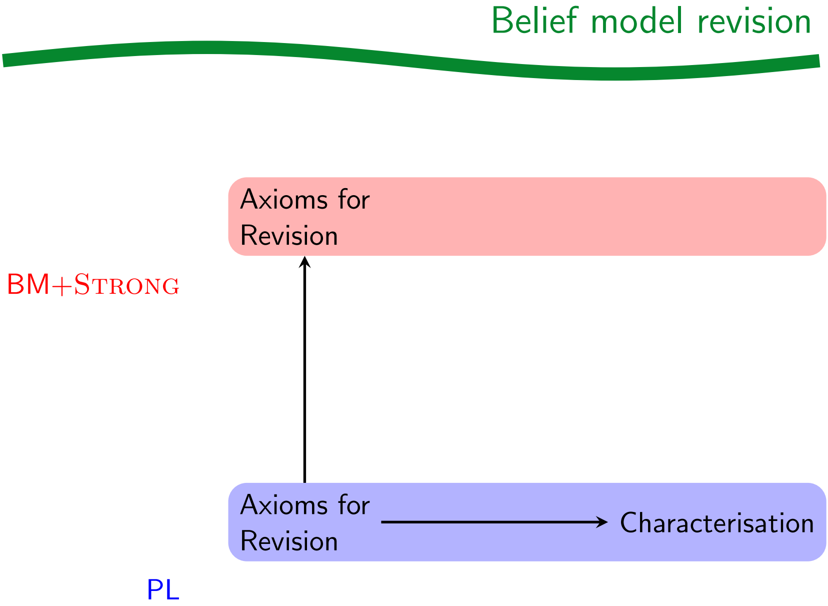
Interestingly, contraction seems more recalcitrant: de Cooman does not provide a "belief structure" version of contraction.

Belief model revision



PL


Belief model revision
Axioms for Revision
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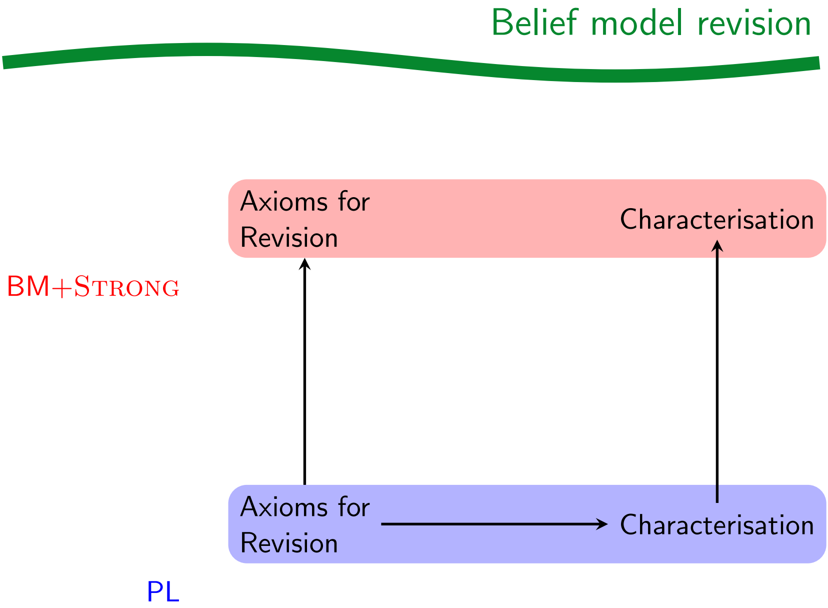
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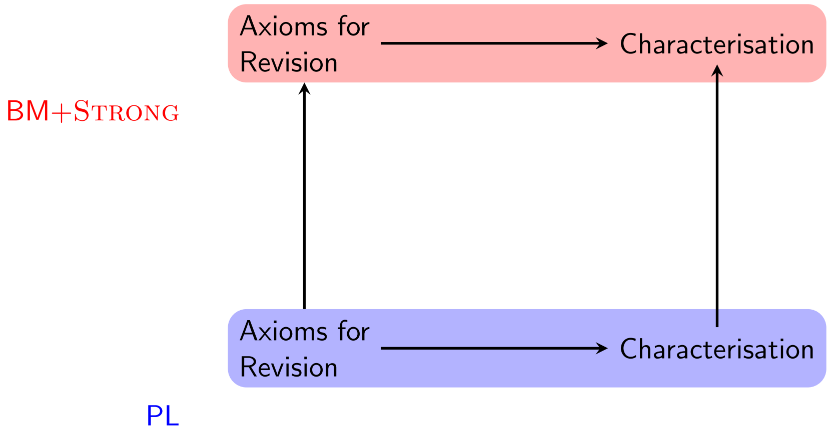

A further property

In what follows we will also need the following property:

$$\text{For distinct } a, b, c \in \mathbf{M}, c \not\leq a \wedge b \qquad (*)$$

This is a property that all distributive lattices satisfy, but I suspect this property is weaker than distributivity.

Merge: the basic idea

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We want a function that maps Ψ to some belief set, subject to some constraints:

- It must satisfy some independent constraints (including consistency)
- It must be "as close" to the opinions of the members of Ψ as possible
- It must treat the different members of Ψ "fairly"

How to make a merging operator

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One way is to construct a Δ on the basis of a sort of "entrenchment relation" over $\mathbf{M}$.

Alternatively, you can construct a Δ using a "distance" over $\mathbf{M}$ and a method of aggregating distances.

Aside: a relation to AGM

If Δ is a merging operator, then define $K_\mu^* = \Delta_\mu(K)$. This is AGM revision.

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The aggregate by minimising that distance.

Belief models make new knowledge

Axioms for
BM + Specifics ⟶ Results

BM(+...)

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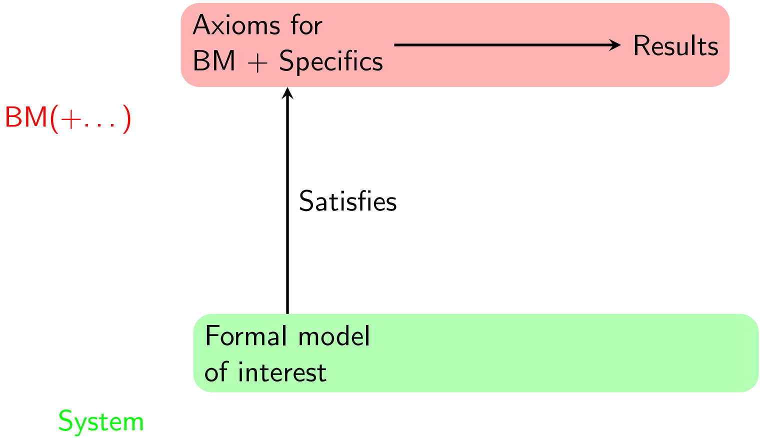

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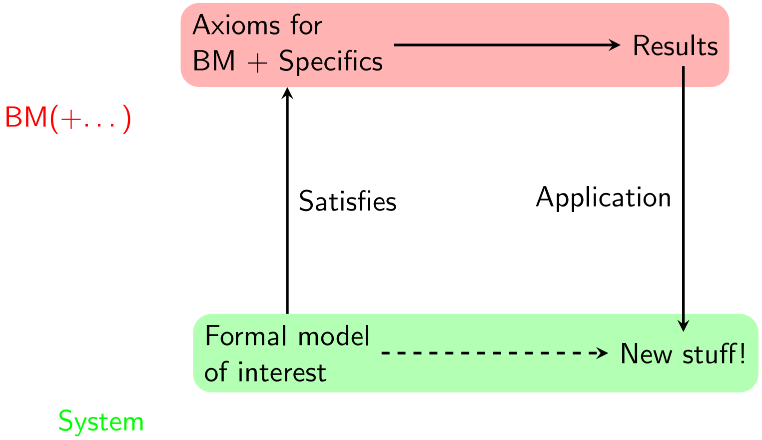

A worked example

Start with the so-called "drastic distance":

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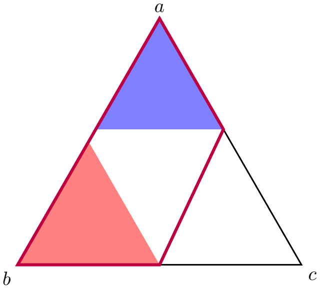
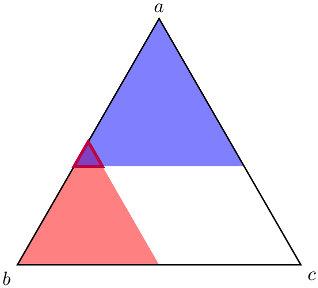
$D_d(w, \Psi) = \sum D_d(w, \phi) =$ The number of $\phi \in \Psi$ that w is not in.

Then we minimise that: meaning, we pick the maximal (w.r.t cardinality) consistent subsets.

Discontinuous merging?





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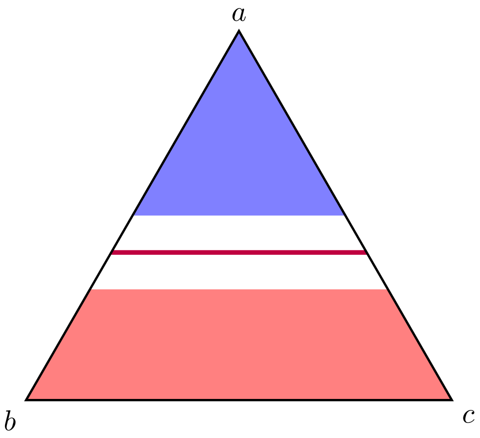
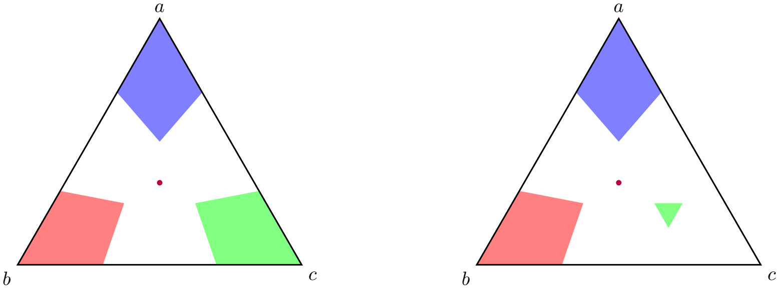
This often yields aggregation more "precise" than you might want.

Weird precision?



Respect imprecision



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For Euclidean distance: you get unweighted linear pooling.

Open questions

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- What about impossibility theorems?
- How weak is the additional property? Can we weaken "strongness" to something something infima of maximal ideals?

Summary

- Belief structures gives us a great way to easily import and generalise a bunch of work done using propositional logic
- More generally, it's remarkable how rich an interesting a theory of rational attitudes we can extract from just the concepts of Informativeness, Coherence and Closeness.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 792292.

Bonus material

- AGM expansion, translated
- Merging operator
- Syncretic assignment

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1. K_A^+ is a belief set (i.e. closed under entailment and consistent)

Belief models

Call $E(b, c)$ the expansion operator for learning c on having beliefs b.

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6. For all K and A, K_A^+ is the smallest belief set satisfying the above conditions

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5. If $b \preceq d$ then $E(b, c) \preceq E(d, c)$
6. $E(b, -)$ is the least informative of all the operators satisfying the above

Representation

AGM

If K_A^+ satisfies the above conditions, then $K_A^+ = Cn(K \cup \{A\})$.

Belief models

If E satisfies the above, then $E(b, c) = Cl_{\mathbf{S}}(\sup\{b, c\})$.

Back

Call $\Delta(\Psi, \mu)$ – or $\Delta_\mu(\Psi)$ – a *merging operator* if Ψ is a multiset of belief models, and μ is a belief model representing the constraints the aggregate belief must satisfy, and Δ satisfies:

- $\mu \preceq \Delta_\mu(\Psi)$
- If μ is consistent then $\Delta_\mu(\Psi)$ is consistent
- If $\bigvee \Psi \vee \mu$ is consistent then $\Delta_\mu(\Psi) = \bigvee \Psi \vee \mu$
- If $\mu \preceq \phi_1$ and $\mu \preceq \phi_2$ then $\Delta_\mu(\phi_1 \sqcup \phi_2) \vee \phi_1$ is consistent if and only if $\Delta_\mu(\phi_1 \sqcup \phi_2) \vee \phi_2$
- $\Delta_\mu(\Psi_1 \sqcup \Psi_2) \preceq \Delta_\mu(\Psi_1) \vee \Delta_\mu(\Psi_2)$
- If $\Delta_\mu(\Psi) \vee \Delta_\mu(\Psi_2)$ is consistent then, $\Delta_\mu(\Psi_1) \vee \Delta_\mu(\Psi_2) \preceq \Delta_\mu(\Psi_1 \sqcup \Psi_2)$
- $\Delta_{\mu_1 \vee \mu_2}(\psi) \preceq \Delta_{\mu_1}(\Psi) \vee \mu_2$
- If $\Delta_{\mu_1}(\Psi) \vee \mu_2$ is consistent then $\Delta_{\mu_1}(\Psi) \vee \mu_2 \preceq \Delta_{\mu_1 \vee \mu_2}(\psi)$

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Syncretic assignments

A *syncretic assignment* is an assignment of a total preorder $\trianglelefteq_\Psi$ to each multiset Ψ, such that:

- For each Ψ, $\trianglelefteq_\Psi$ is a total order on $\mathbf{M}$
- If $a \in M(\bigvee \Psi)$ and $b \in M(\bigvee \Psi)$ then $a \trianglelefteq_\Psi b$
- If $a \in M(\bigvee \Psi)$ but $b \notin M(\bigvee \Psi)$ then $a \triangleleft_\Psi b$
- For all $a \in M(\phi)$ there is some $b \in M(\phi')$ such that $b \trianglelefteq_{\phi \sqcup \phi'} a$
- If $a \trianglelefteq_{\Psi_1} b$ and $a \trianglelefteq_{\Psi_2} b$ then $a \trianglelefteq_{\Psi_1 \sqcup \Psi_2} b$
- If $a \triangleleft_{\Psi_1} b$ and $a \trianglelefteq_{\Psi_2} b$ then $a \triangleleft_{\Psi_1 \sqcup \Psi_2} b$
- $\trianglelefteq_\Psi$ is *smooth*, meaning for all μ, for all $m \in M(\mu)$, if m is not minimal with respect to $\trianglelefteq_\Psi$ then there is an $m' \in M(\mu)$ such that m' is minimal and $m' \triangleleft_\Psi m$.

Δ is a merging operator iff there is a syncretic assignment such that $\Delta_\mu(\Psi) = \inf_{\preceq} \min_{\trianglelefteq_\Psi}\{M(\mu)\}$.

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