Belief models
A very general theory of aggregation

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My plan is to show how far we can get with just these abstract ideas.
The very general theory of “Belief Models”\(^1\) provides a neat generalisation of (part of) AGM belief revision theory.

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The very general theory of “Belief Models”\textsuperscript{1} provides a neat generalisation of (part of) AGM belief revision theory.

My plan is to show that the same sort of generalisation can be applied to “merging operators”\textsuperscript{2} for aggregating (propositional) knowledge bases.


Belief models

The recipe
  AGM expansion
  AGM revision
  Merging operators

Cooking up aggregation rules
Belief models

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   AGM revision
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Cooking up aggregation rules
Consider the structure of sets of sentences of a propositional logic.

**Ordering** Sets of sentences are (partially) ordered by the subset relation.
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**Lattice structure** For any pair of sets of sentences $A, B$, there is a set of sentences that is the least upper bound $A \lor B$, and another that is greatest lower bound $A \land B$. 
Some facts about sets of sentences

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**Top** The set of all sentences – the top of the ordering – is not coherent.
Lower previsions provide a general model of uncertainty. They are a generalisation of probability theory.
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Lower probabilities (lower previsions restricted to events) are superadditive but not necessarily additive: $L(X \text{ or } Y) \geq L(X) + L(Y)$ for incompatible $X, Y$. 
Some facts about lower previsions

Ordering Lower previsions are partially ordered by pointwise dominance. \( L \preceq L' \) iff for all \( x \), \( L(x) \leq L'(x) \).
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Top  The lower prevision that assigns $\infty$ to all gambles – the top of the structure – is not coherent.
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In particular, $1_S \not\in C$. 
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In particular, $1_\mathcal{S} \not\in \mathcal{C}$.

$\langle \mathcal{S}, \mathcal{C}, \preceq \rangle$ is called a belief structure.
Lattice structure

\[
\begin{align*}
\text{lattice structure} & \\
\downarrow & \\
& \text{representation}
\end{align*}
\]
Let $\overline{C} = C \cup \{1_s\}$, and define:

$$\operatorname{Cl}_S(b) = \inf\{c \in \overline{C}, b \preceq c\}$$
Closure for sets of sentences

\{ A, B, C, A \land B, \neg (A \land B) \rightarrow A \land B, \ldots \} 

\{ A, B, A \land B, \neg (A \land B) \rightarrow A \land B, \ldots \} 

\{ A, B \} 

\{ \neg (A \land B) \rightarrow A \land B \} 

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Examples of belief structures

- Propositional logic (with $\subseteq$, and $Cn$)
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- Sets of desirable gambles, choice functions...
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▶ Sets of desirable gambles, choice functions...
▶ Preference relations, comparative confidence relations?
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We have a propositional logic $\mathcal{L}$, and use a set of sentences $K$ to represent the beliefs of an agent. The agent beliefs $X \in \mathcal{L}$ just in case $X \in K$. Of particular interest are those agents whose belief set $K$ is consistent, and closed under entailment. We can provide some axioms for straightforward learning $A$ given belief set $K$, such that $K + A$ can be characterised.
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We can provide some axioms for straightforward learning $A$ given belief set $K$, such that $K_A^+$ can be characterised.
Belief model expansion

Axioms for Expansion  \rightarrow  Characterisation

PL
Belief model expansion

Axioms for Expansion

BM

Axioms for Expansion

PL

Characterisation
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Axioms for Expansion \rightarrow Characterisation
This recipe is quite generalisable: take a result framed in the theory of propositional logic, and (if you’re lucky) it will also hold in some version of the belief models framework.
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Consider the maximal consistent sets of sentences for a propositional logic. We can identify these with the set of states.
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Consider the maximal consistent sets of sentences for a propositional logic. We can identify these with the set of states.

Let \( M = \{ m \in C : \text{For all } c \in C, m \preceq c \Rightarrow m = c \} \)

Call a belief structure a \textit{strong} belief structure, when, for all \( c \in C \),
\( c = \inf \{ m \in M, c \preceq m \} \).
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Lattice structure

\[ a \rightarrow b \quad a \leftrightarrow b' \quad a' \rightarrow b \quad a' \leftrightarrow b' \]

\[ a \text{ or } b \quad b \rightarrow a \quad a \rightarrow b \quad a' \text{ or } b' \]
For strong belief structures, we can do for AGM revision what we just did for expansion!
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Interestingly, contraction seems more recalcitrant: de Cooman does not provide a “belief structure” version of contraction.
Belief model revision

Axioms for Revision

Characterisation

PL
Belief model revision

Axioms for Revision

BM+Strong

Axioms for Revision

PL

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Axioms for Revision → Characterisation

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Cooking up aggregation rules
In what follows we will also need the following property:

For distinct \( a, b, c \in M \), \( c \nsubseteq a \land b \) \hspace{1cm} (*)

This is a property that all distributive lattices satisfy, but I suspect this property is weaker than distributivity.
Say you have a group of people, each with their own – possibly conflicting – beliefs. How best to aggregate their beliefs?
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Consider a multiset $\Psi$ of belief models.

We want a function that maps $\Psi$ to some belief set, subject to some constraints:

- It must satisfy some independent constraints (including consistency)
- It must be “as close” to the opinions of the members of $\Psi$ as possible
- It must treat the different members of $\Psi$ “fairly”
The (propositional logic) literature on merging operators provides two main ways to develop a merging operator.
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One way is to construct a $\Delta$ on the basis of a sort of “entrenchment relation” over $M$.

Alternatively, you can construct a $\Delta$ using a “distance” over $M$ and a method of aggregating distances.
If $\Delta$ is a merging operator, then define $K_\mu^* = \Delta_\mu(K)$. This is AGM revision.
One approach to constructing merging operators is to start from a distance between maximal belief models: $D(w, w')$. 

Define a distance between worlds and belief sets:

$$D(w, \phi) = \min_{\phi \preceq w'} \{ D(w, w') \}$$

Define a distance between worlds and multisets of belief sets:

$$D(w, \Psi) = \sum_{\phi \in \Psi} D(w, \phi)$$

The aggregate by minimising that distance.
Distance based merging

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Belief models make new knowledge

Axioms for BM + Specifics → Results

BM(+...)

Belief models make new knowledge

Axioms for BM + Specifics → Results

Satisfies

Formal model of interest

System

BM(+)
Belief models make new knowledge

Axioms for BM + Specifics \rightarrow Results

Satisfies

Formal model of interest \rightarrow New stuff!

System
A worked example

Start with the so-called “drastic distance”:

\[
D_d(w, w') = \begin{cases} 
0 & \text{if } w = w' \\
1 & \text{otherwise}
\end{cases}
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Then we minimise that: meaning, we pick the maximal (w.r.t cardinality) consistent subsets.
A worked example

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\[ D_d(w, \Psi) = \sum D_d(w, \phi) = \text{The number of } \phi \in \Psi \text{ that } w \text{ is not in.} \]

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Discontinuous merging?
Other ways to merge

What if we use, say, Euclidean distance rather than drastic distance?
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Other ways to merge

What if we use, say, Euclidean distance rather than drastic distance?

Then we are minimising the sum of minimum distances.

This often yields aggregation more “precise” than you might want.
Weird precision?
Respect imprecision
What happens to precise input?

What if each lower prevision in $\Psi$ is, in fact, a linear prevision (i.e. a precise probability)?
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For Euclidean distance: you get unweighted linear pooling.
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Open questions

- Convex combinations of coherent lower previsions are coherent, so how about just aggregate by linear pooling?
- What about other distances? Or distance aggregation other than $\sum$?
- What about impossibility theorems?
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What about other distances? Or distance aggregation other than \( \sum \)?

What about impossibility theorems?

How weak is the additional property? Can we weaken “strongness” to something infima of maximal ideals?
Belief structures gives us a great way to easily import and generalise a bunch of work done using propositional logic.

More generally, it’s remarkable how rich and interesting a theory of rational attitudes we can extract from just the concepts of Informativeness, Coherence and Closeness.

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Bonus material

- AGM expansion, translated
- Merging operator
- Syncretic assignment
Axioms

<table>
<thead>
<tr>
<th>AGM</th>
<th>Belief models</th>
</tr>
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<tbody>
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<td>Call $K_A^+$ the expansion of $K$ by (consistent) $A$.</td>
<td>Call $E(b, c)$ the expansion operator for learning $c$ on having beliefs $b$.</td>
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### AGM

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2. $A \in K_A^+$

### Belief models

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2. $c \preceq E(b, c)$
### Axioms

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3. $K \subseteq K^+_A$

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5. If $K \subseteq H$ then $K_A^+ \subseteq H_A^+$

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5. If $b \preceq d$ then $E(b, c) \preceq E(d, c)$
**Axioms**

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2. $A \in K_A^+$
3. $K \subseteq K_A^+$
4. If $A \in K$ then $K_A^+ = K$
5. If $K \subseteq H$ then $K_A^+ \subseteq H_A^+$
6. For all $K$ and $A$, $K_A^+$ is the smallest belief set satisfying the above conditions

**Belief models**

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3. $b \preceq E(b, c)$
4. If $c \preceq b$ then $E(b, c) = b$
5. If $b \preceq d$ then $E(b, c) \preceq E(d, c)$
6. $E(b, -)$ is the least informative of all the operators satisfying the above
AGM
If $K_A^+$ satisfies the above conditions, then
$$K_A^+ = Cn(K \cup \{A\}).$$

Belief models
If $E$ satisfies the above, then
$$E(b, c) = Cl_S(\sup\{b, c\}).$$
Call $\Delta(\Psi, \mu)$ – or $\Delta_\mu(\Psi)$ – a merging operator if $\Psi$ is a multiset of belief models, and $\mu$ is a belief model representing the constraints the aggregate belief must satisfy, and $\Delta$ satisfies:

- $\mu \preceq \Delta_\mu(\Psi)$
- If $\mu$ is consistent then $\Delta_\mu(\Psi)$ is consistent
- If $\bigvee \Psi \vee \mu$ is consistent then $\Delta_\mu(\Psi) = \bigvee \Psi \vee \mu$
- If $\mu \preceq \phi_1$ and $\mu \preceq \phi_2$ then $\Delta_\mu(\phi_1 \sqcup \phi_2) \vee \phi_1$ is consistent if and only if $\Delta_\mu(\phi_1 \sqcup \phi_2) \vee \phi_2$
- $\Delta_\mu(\Psi_1 \sqcup \Psi_2) \preceq \Delta_\mu(\Psi_1) \vee \Delta_\mu(\Psi_2)$
- If $\Delta_\mu(\Psi) \vee \Delta_\mu(\Psi_2)$ is consistent then,
  $\Delta_\mu(\Psi_1) \vee \Delta_\mu(\Psi_2) \preceq \Delta_\mu(\Psi_1 \sqcup \Psi_2)$
- $\Delta_{\mu_1 \vee \mu_2}(\psi) \preceq \Delta_{\mu_1}(\psi) \vee \mu_2$
- If $\Delta_{\mu_1}(\Psi) \vee \mu_2$ is consistent then $\Delta_{\mu_1}(\Psi) \vee \mu_2 \preceq \Delta_{\mu_1 \vee \mu_2}(\psi)$
A *syncretic assignment* is an assignment of a total preorder \( \trianglelefteq_\Psi \) to each multiset \( \Psi \), such that:

- For each \( \Psi \), \( \trianglelefteq_\Psi \) is a total order on \( M \).
- If \( a \in M(\bigvee \Psi) \) and \( b \in M(\bigvee \Psi) \) then \( a \trianglelefteq_\Psi b \).
- If \( a \in M(\bigvee \Psi) \) but \( b \not\in M(\bigvee \Psi) \) then \( a \triangleleft_\Psi b \).
- For all \( a \in M(\phi) \) there is some \( b \in M(\phi') \) such that \( b \trianglelefteq_{\phi \sqcup \phi'} a \).
- If \( a \trianglelefteq_{\Psi_1} b \) and \( a \trianglelefteq_{\Psi_2} b \) then \( a \trianglelefteq_{\Psi_1 \sqcup \Psi_2} b \).
- If \( a \triangleleft_{\Psi_1} b \) and \( a \triangleleft_{\Psi_2} b \) then \( a \triangleleft_{\Psi_1 \sqcup \Psi_2} b \).
- \( \trianglelefteq_\Psi \) is *smooth*, meaning for all \( \mu \), for all \( m \in M(\mu) \), if \( m \) is not minimal with respect to \( \trianglelefteq_\Psi \) then there is an \( m' \in M(\mu) \) such that \( m' \) is minimal and \( m' \triangleleft_\Psi m \).

\( \Delta \) is a merging operator iff there is a syncretic assignment such that \( \Delta_\mu(\Psi) = \inf_{\leq} \min_{\trianglelefteq_\Psi} \{ M(\mu) \} \).