Aggregation for Belief Models

Seamus Bradley

University of Leeds/TiLPS

October 8, 2018

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This gives us a new insight into how to aggregate the opinions of probabilistic agents.

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AGM generalised

Belief merge for propositions

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Of particular interest are those agents whose belief set K is consistent, and closed under entailment.

AGM Expansion

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6. For all K and A, K^+_A is the smallest belief set satisfying the above conditions

Representation

If K_A^+ satisfies the above conditions, then $K_A^+ = Cn(K \cup \{A\})$.

Some notes

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So: can we define "expansion" for different representations of belief states?

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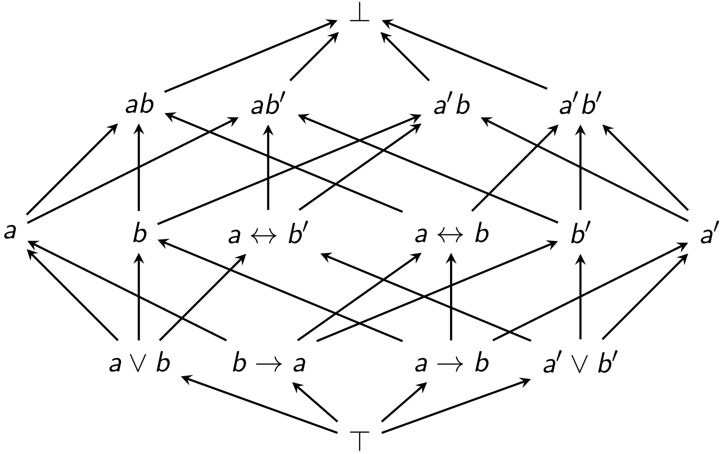
$\langle \mathbf{S}, \mathbf{C}, \preceq \rangle$ is called a *belief structure*.

Lattice structure



Closure

Let $\overline{\mathbf{C}} = \mathbf{C} \cup \{1_{\mathbf{S}}\}$, and define:

$$Cl_{\mathbf{S}}(b) = \inf\{c \in \overline{\mathbf{C}}, b \preceq c\}$$

Closure for sets of sentences

$$\{A, B, C, A \land B, \neg(A \land B) \rightarrow A \land B, \dots\}$$

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- Ranking functions
- Sets of desirable gambles, choice functions...
- Preference relations, comparative confidence relations?

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Belief models

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6. $E(b, -)$ is the least informative of all the operators satisfying the above

Representation

AGM

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Belief models

If E satisfies the above, then $E(b, c) = Cl_{\mathbf{S}}(\sup\{b, c\})$.

Strong belief structures

Consider the maximal consistent sets of sentences for a propositional logic. We can identify these with the set of states.

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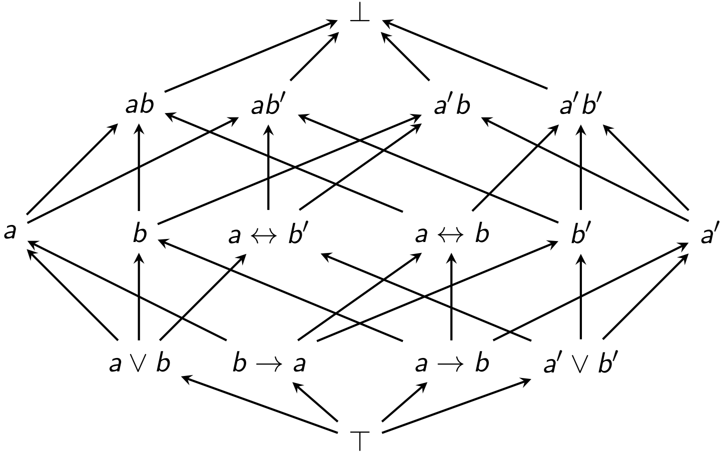
Let $\mathbf{M} = \{m \in \mathbf{C} : \text{For all } c \in \mathbf{C}, m \preceq c \Rightarrow m = c\}$

Call a belief structure a *strong* belief structure, when, for all $c \in \mathbf{C}$, $c = \inf\{m \in \mathbf{M}, c \preceq m\}$.

Lattice structure



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Interestingly, contraction seems more recalcitrant: de Cooman does not provide a "belief structure" version of contraction.

AGM generalised

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The basic idea

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Say you have a group of people, each with their own – possibly conflicting – beliefs. How best to aggregate their beliefs?

Consider a multiset Ψ of belief sets.

We want a function that maps Ψ to some belief set, subject to some constraints:

- It must satisfy some independent constraints (including consistency)
- It must be "as close" to the opinions of the members of Ψ as possible
- It must treat the different members of Ψ "fairly"

Merging operators

Call $\Delta(\Psi, \mu)$ – or $\Delta_\mu(\Psi)$ – a *merging operator* if Ψ is a multiset of belief sets, and μ is a belief set representing the constraints the aggregate belief must satisfy, and Δ satisfies:

1. $\Delta_\mu(\Psi) \vdash \mu$

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4. ...

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Aside: a relation to AGM

If Δ is a merging operator, then define $K^*_\mu = \Delta_\mu(K)$. This is AGM revision.

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- Probabilities: McConway's theorem

Initial remarks on limitative theorems

- Note that typically what is required from aggregation is a maximially informative output (total order, maximal belief set, precise probability), and relaxing this condition is a natural way to avoid the limitative theorems.
- Many theorems also mention some sort of "complexity of agenda" requirement, which is hard to express in the belief models framework.

Summary

- The Belief Structures approach suggests a natural (and fairly mechanical) way of producing new results about probabilities by generalising results involving propositional logic
- The cost is that it is more natural to do things in terms of lower probabilities, rather than precise probabilities.
- (I don't actually think this is a "cost" at all, but that's another talk)

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2. If μ is consistent then $\Delta_\mu(\Psi)$ is consistent
3. If $\bigwedge \Psi$ is consistent with μ then $\Delta_\mu(\Psi) = \bigwedge \Psi \wedge \mu$
4. If $\phi \vdash \mu$ and $\phi' \vdash \mu$ then $\Delta_\mu(\phi \sqcup \phi') \wedge \phi$ is consistent iff $\Delta_\mu(\phi \sqcup \phi') \wedge \phi'$ is
5. $\Delta_\mu(\Psi_1) \wedge \Delta_\mu(\Psi_2) \vdash \Delta_\mu(\Psi_1 \sqcup \Psi_2)$
6. If $\Delta_\mu(\Psi) \wedge \Delta_\mu(\Psi_2)$ is consistent then, $\Delta_\mu(\Psi_1 \sqcup \Psi_2) \vdash \Delta_\mu(\Psi_1) \wedge \Delta_\mu(\Psi_2)$
7. $\Delta_{\mu_1}(\Psi) \wedge \mu_2 \vdash \Delta_{\mu_1 \wedge \mu_2}(\psi)$
8. If $\Delta_{\mu_1}(\Psi) \wedge \mu_2$ is consistent then $\Delta_{\mu_1 \wedge \mu_2}(\psi) \vdash \Delta_{\mu_1}(\Psi) \wedge \mu_2$

Back

Syncretic assignments

A *syncretic assignment* is an assignment of a total preorder $\trianglelefteq_\Psi$ to each multiset Ψ, such that:

1. If $a \models \Psi$ and $b \models \Psi$ then $a \trianglelefteq_\Psi b$
2. If $a \models \Psi$ but $b \not\models \Psi$ then $a \triangleleft_\Psi b$
3. If $\Psi_1 \leftrightarrow \Psi_2$, then $\trianglelefteq_{\Psi_1} = \trianglelefteq_{\Psi_2}$
4. For all $a \models \phi$ there is some $b \models \phi'$ such that $b \trianglelefteq_{\phi \sqcup \phi'} a$
5. If $a \trianglelefteq_{\Psi_1} b$ and $a \trianglelefteq_{\Psi_2} b$ then $a \trianglelefteq_{\Psi_1 \sqcup \Psi_2} b$
6. If $a \triangleleft_{\Psi_1} b$ and $a \trianglelefteq_{\Psi_2} b$ then $a \triangleleft_{\Psi_1 \sqcup \Psi_2} b$

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